

# Using Mathcad to Solve Laplace Transforms

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### Introduction

Using Laplace transforms is a common method of solving linear systems of differential equations with initial conditions. Such systems occur frequently in control theory, circuit design, and other engineering applications. The main advantage of using Laplace transforms is that the solution of the differential equations is reduced to algebraic manipulation. The initial conversion from the time domain to the Laplace domain and the deconvolution of the Laplace domain solution back to the time domain however can be challenging and time-consuming. Mathcad provides a simple and powerful method for dealing with these operations. Successfully using Mathcad however does require that some of the transformations to the Laplace domain be done manually, so you will need a few simple transforms that are shown in the table below. On the other hand, obtaining the Laplace transforms for simultaneous equations and conversion of  $F(s)$  back to the time domain are both simple operations.

**Table 1**  
**Useful Laplace Transforms**

$f(t)$	$\mathcal{L}\{F\}$
$u(t-a)$ $u = 1$ when $t-a > 0$ Unit step $u = 0$ when $t-a < 0$	$\frac{e^{-as}}{s}$
$u(t-a)f(t-a)$	$e^{-as}F(s)$
$\delta(t-a) = \begin{cases} 1 & t = a \\ 0 & t \neq a \end{cases}$ Dirac delta	$e^{-as}$
$\frac{df(t)}{dt}$	$sF(s) - f(0)$
$\frac{d^2f(t)}{dt^2}$	$s^2F(s) - sf(0) - f'(0)$
$\int_0^t f(\tau)d\tau$	$\frac{1}{s}F(s)$

In this set of notes you will solve first a ordinary differential equation using Laplace transforms. In the second example you will solve a system of differential equations also using Laplace transforms and Mathcad's ability to symbolically solve systems of equations.

The first equation we will solve is

$$y'' + 2y' + 2y = 5u(t - 2\pi)\sin(t)$$

Where  $u(\tau) = 0$  when  $\tau < 0$  and  $u(\tau) = 1$  when  $\tau > 0$   
with the boundary conditions  $y(0) = 1, y'(0) = 0$ . (Advanced Engineering Mathematics  
eight edition, Erwin Kreyszig, problem 23, problem set 5.5, page 283)

Note that Mathcad uses the symbol  $\Phi(t)$  to represent the unit step function  $u(t)$

The system of differential equations we will solve is

$$\begin{aligned}y_1' &= 2y_1 - 4y_2 + u(t-1)e^t \\ y_2' &= y_1 - 3y_2 + u(t-1)e^t\end{aligned}$$

with the boundary conditions  $y_1(0) = 3, y_2(0) = 0$  (Advanced Engineering Mathematics  
eight edition, Erwin Kreyszig, problem set 5.6, no. 13, page 295)

### I. Solving a Single Equation

You will now solve the differential equation

$$y'' + 2y' + 2y = 5u(t - 2\pi)\sin(t)$$

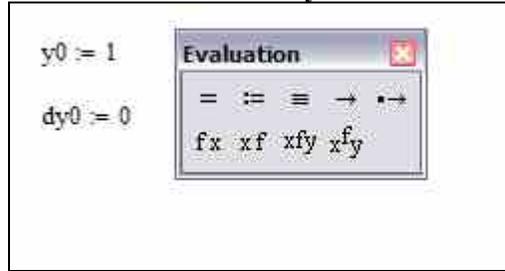
Where  $u(\tau) = 0$  when  $\tau < 0$  and  $u(\tau) = 1$  when  $\tau > 0$   
with the boundary conditions  $y(0) = 0, y'(0) = 1$ . (note 23 page 283)

Using Laplace transforms to solve this differential equation requires first converting the equation to the Laplace domain, then solving the resulting transform for  $Y(s)$  and finally, deconvolving that function back into the time domain. Unfortunately, Mathcad has difficulty converting entire equations to the Laplace domain. Therefore it is best to transform an equation into the Laplace domain on a term by term basis. Converting each derivative or function to the Laplace domain individually, using Mathcad to find the function  $F(s)$  for any functions  $f(t)$  and writing the final equation. The Laplace transforms of the functions  $Y(s)$  are found using Mathcad's solve blocks. Finally, you will use Mathcad's inverse Laplace transform capability to deconvolve the  $Y(s)$  into  $y(t)$ .

#### Enter the Boundary Conditions and f(T)

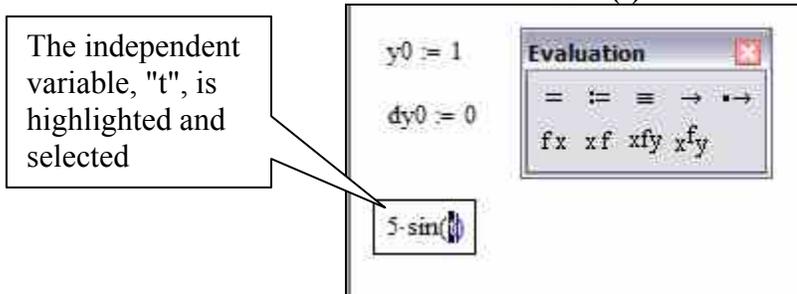
1. Open Mathcad in the usual manner. Begin by entering the boundary conditions. Define variables  $y_0$  and  $dy_0$  that contain the values of  $y(0)$  and  $y'(0)$ . Your worksheet should look like figure 1. Remember to open the evaluation toolbar by selecting "View/Toolbars/Evaluation" from the menu and pressing the "[:=" button. Alternatively you may press the "[:=" key to create the "[:=" symbol.

**Figure 1**  
**Enter the Boundary Conditions**



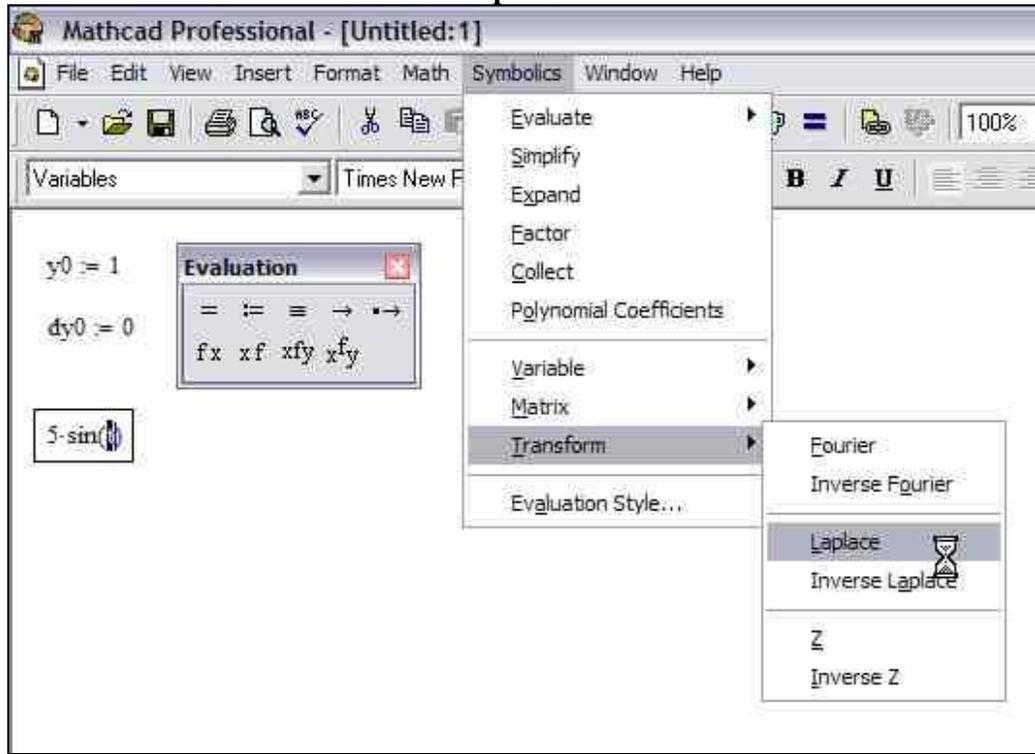
- Mathcad can transform most functions of an independent variable, typically “t” for time, into the Laplace domain without much difficulty. Time shifted functions, such as  $u(t-a)f(t-a)$ , need to be treated in two steps. First, obtain the transform of  $f(t-a)$  using Mathcad's Laplace transform option. This requires defining a new variable  $T=t-a$ . The the function is rewritten in terms of the new r variable by substituting  $T+a$  for  $t$ . Then, manually multiply the results by  $e^{-as}$ . In this example, enter the function “5\*sin(t)” just below the boundary conditions. Remember to insert and multiplication symbol “\*” between the five and the letter “s”. Note that the time shift for this function is  $2\pi$  and that  $\sin(t) = \sin(t + 2\pi)$  for all  $t > 2\pi$ . When you have finished entering the function, highlight the letter “t” so that it displays as white on black as shown in figure 2. You can highlight the letter by moving the mouse cursor just above and to the left of the letter, pressing the left mouse button and moving the mouse to the lower right of the letter.

**Figure 2**  
**Enter f(t)**

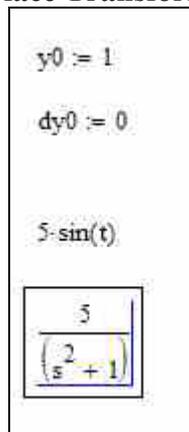


- Choose "Symbolics/Transform/Laplace" from the menu option as shown in figure 3 a. This option must know which variable in the function is the independent variable. You indicate the name of the independent variable by highlighting that variable before you select the function. After you select the Laplace transform option, the menu will close and the mouse cursor will turn into a light bulb. After a few seconds, Mathcad will display the Laplace transform of the function just below it. After the completed operation your screen should look like figure 3b

**Figure 3 a**  
**Menu for Laplace Transforms**



**Figure 3 B**  
**the Laplace Transform of f(T)**



### Writing F(s)

- You are now ready to write the Laplace transform of the original differential equation. Recall that the original differential equation and its Laplace transform are

$$y'' + 2y' + 2y = 5u(t - 2\pi)\sin(t)$$

$$(s^2 Y(s) - sy(0) - y'(0)) + 2(sY(s) - y(0)) = L(f(t))$$

Where  $L(f(t)) =$  the Laplace transform of the function of  $t$  on the right hand side. Move the red cross cursor some distance below the last entry on your worksheet. You will now enter the left-hand side of the Laplace transform of the original differential equation. It is much easier to do this portion of the solution manually and have Mathcad perform the symbolic manipulation necessary to solve for  $Y(s)$ . Use the symbol “Y” to represent  $Y(s)$  and enter the left-hand side onto the worksheet. Remember to use the “^” key to write a superscript. Also use the names of the variables you created earlier for  $y(0)$  and  $y'(0)$ . After you've entered the left-hand side of the Laplace transform the equation should look something like Figure 4.

**Figure 4**  
**Entering the Laplace Transforms of Y(s) and Its Derivatives**

$$s^2 \cdot Y - s \cdot y0 - dy0 + 2 \cdot (s \cdot Y - y0) + 2 \cdot Y$$

- To finish writing this equation you will use the Boolean “=”. Choose “View/Toolbars/Boolean” from the menu to create the toolbar that contains the “=” button. The Time shifted function  $u(t-2\pi)$  becomes  $e^{-2\pi s}$  in the Laplace domain. The second term in this function,  $\sin(t)$ , is easy to time shift. You can copy the Laplace transform you obtained earlier by simply highlighting the transform so that it appears reversed (black background white letters). Then press “Ctrl C” to copy the transform into the clipboard. At this point your worksheet should resemble figure 5.

**Figure 5**  
**Copying the Laplace Transform of f(T)**

$y0 := 1$   
 $dy0 := 0$   
 $5 \cdot \sin(t)$

Highlight the Laplace transform by dragging the mouse cursor over it. Copy it to the clipboard by pressing “Ctrl C”

The Laplace transform of the  $2\pi$  delay is  $e^{-2\pi s}$

The Laplace transform will be copied here

$$s^2 \cdot Y - s \cdot y0 - dy0 + 2 \cdot (s \cdot Y - y0) + 2 \cdot Y = e^{-2\pi \cdot s}$$

- Copy the transfer function into the Laplace transform of the differential equation. Move the mouse cursor to the black rectangle placeholder in the Laplace transform and press the left mouse button. This will highlight the place holder and cause the blue “inverted L” cursor to appear a round of the black rectangle.

Press "Ctrl V". The Laplace function should be copied into the finished equation. Your worksheet should now look like figure 6.

**Figure 6**  
**The Finished Laplace Transform**

$y(0) := 1$   
 $dy(0) := 0$   
 $5 \cdot \sin(t)$   
 $\frac{5}{(s^2 + 1)}$

Check that the entire Transform has been copied.

$s^2 \cdot Y - s \cdot y(0) - dy(0) + 2 \cdot (s \cdot Y - y(0)) + 2 \cdot Y = e^{-2 \cdot \pi \cdot s} \cdot \frac{5}{(s^2 + 1)}$

7. You will now use Mathcad's ability to do algebra to solve the equation for Y. Make this equation part of a "solve block" by typing the word "given" directly above the equation. Afterwards move the mouse cursor directly below the equation and type "find (Y) →". To make the arrow operator appear, choose "View/Toolbars/Evaluation". Use the  button and press "Enter". The mouse cursor should change to a light bulb. After a while the Laplace transform for Y(s) should appear to the right of the arrow. Your worksheet should look like figure 7.

**Figure 7**  
**Using Mathcad to Solve for Y (S)**

The word "given" marks the beginning of a Mathcad solve block.

The evaluation toolbar contains arrow operator used in the "find" statement.

The Laplace transform of  $Y(s)$

$y_0 := 1$   
 $dy_0 := 0$   
 $5 \sin(t)$   
 $\frac{5}{(s^2 + 1)}$   
 given  
 $s^2 \cdot Y - s \cdot y_0 - dy_0 + 2 \cdot (s \cdot Y - y_0) + 2 \cdot Y = e^{-2 \cdot \pi \cdot s} \cdot \frac{5}{(s^2 + 1)}$   
 $+ \text{find}(Y) \rightarrow \frac{(s^3 + s + 2 \cdot s^2 + 2 + 5 \cdot \exp(-2 \cdot \pi \cdot s))}{(s^4 + 3 \cdot s^2 + 2 \cdot s^3 + 2 \cdot s + 2)}$

### Converting Back to the Time Domain

8. You are now ready to convert the Laplace transform back into time domain. First, copy the Laplace solution to an area just below. Highlight the entire Laplace transform by first moving the cursor to the upper left-hand corner of the function, pressing the left mouse button, and then dragging the mouse cursor to the lower right hand corner of the function. This will cause the entire function to appear as white characters on black background as shown in figure 8. Then press "Ctrl C" to copy the function into the clipboard.

**Figure 8**  
**Copy the Laplace Transform to the Clipboard**

Place the mouse cursor at the upper left of the function. Press the left mouse button and drag the mouse cursor to the lower right corner.

Then press "Ctrl C" to copy the function to the clipboard.

The screenshot shows a Mathcad worksheet with a Laplace transform result. The result is a fraction:  $\frac{(s^3 + s + 2 \cdot s^2 + 2 + 5 \cdot \exp(-2 \cdot \pi \cdot s))}{(s^4 + 3 \cdot s^2 + 2 \cdot s^3 + 2 \cdot s + 2)}$ . This fraction is highlighted in a black selection box. An 'Evaluation' menu is open, showing options like 'fx', 'xf', 'xfy', and 'x^fy'. A red cross cursor is visible over the selection box.

9. You must copy the Laplace transform to another area in order to return the function to the time domain. Move the mouse cursor to an area just below this given block and press the left mouse button. The red cross cursor will appear. Press "Ctrl V" to copy the Laplace transforming into this area.

**Figure 9**  
**Copy the Laplace Transform to the Worksheet**

$\text{find}(Y) \rightarrow \frac{(s^3 + s + 2 \cdot s^2 + 2 + 5 \cdot \exp(-2 \cdot \pi \cdot s))}{(s^4 + 3 \cdot s^2 + 2 \cdot s^3 + 2 \cdot s + 2)}$

$\frac{(s^3 + s + 2 \cdot s^2 + 2 + 5 \cdot \exp(-2 \cdot \pi \cdot s))}{(s^4 + 3 \cdot s^2 + 2 \cdot s^3 + 2 \cdot s + 2)}$

10. In order to do the deconvolution to the time domain, you must first highlight a letter "s" in the function you just copied so that Mathcad will know what variable name you are using for the Laplace parameter. Highlight any of the letter S. in the function you just copied. Then choose "Symbolics/Transform/Inverse Laplace" from the menu.

**Figure 10**  
**Deconvolving the Laplace Transform**

<p>given</p> $s^2 \cdot Y - s \cdot y_0 - dy_0 + 2 \cdot (s \cdot Y - y_0) + 2 \cdot Y = e^{-2 \cdot \pi \cdot s} \cdot \frac{5}{(s^2 + 1)}$ <p>find(Y) → <math>\frac{(s^3 + s + 2 \cdot s^2 + 2 + 5 \cdot \exp(-2 \cdot \pi \cdot s))}{(s^4 + 3 \cdot s^2 + 2 \cdot s^3 + 2 \cdot s + 2)}</math></p> $\frac{(s^3 + s + 2 \cdot s^2 + 2 + 5 \cdot \exp(-2 \cdot \pi \cdot s))}{(s^4 + 3 \cdot s^2 + 2 \cdot s^3 + 2 \cdot s + 2)}$	
$\exp(-t) \cdot \cos(t) + \exp(-t) \cdot \sin(t) + 2 \cdot \Phi(t - 2 \cdot \pi) \cdot \exp(-t + 2 \cdot \pi) \cdot \cos(t) + \Phi(t - 2 \cdot \pi) \cdot \exp(-t + 2 \cdot \pi) \cdot \sin(t) - 2 \cdot \Phi(t - 2 \cdot \pi) \cdot \cos(t) + \Phi(t - 2 \cdot \pi) \cdot \sin(t)$	

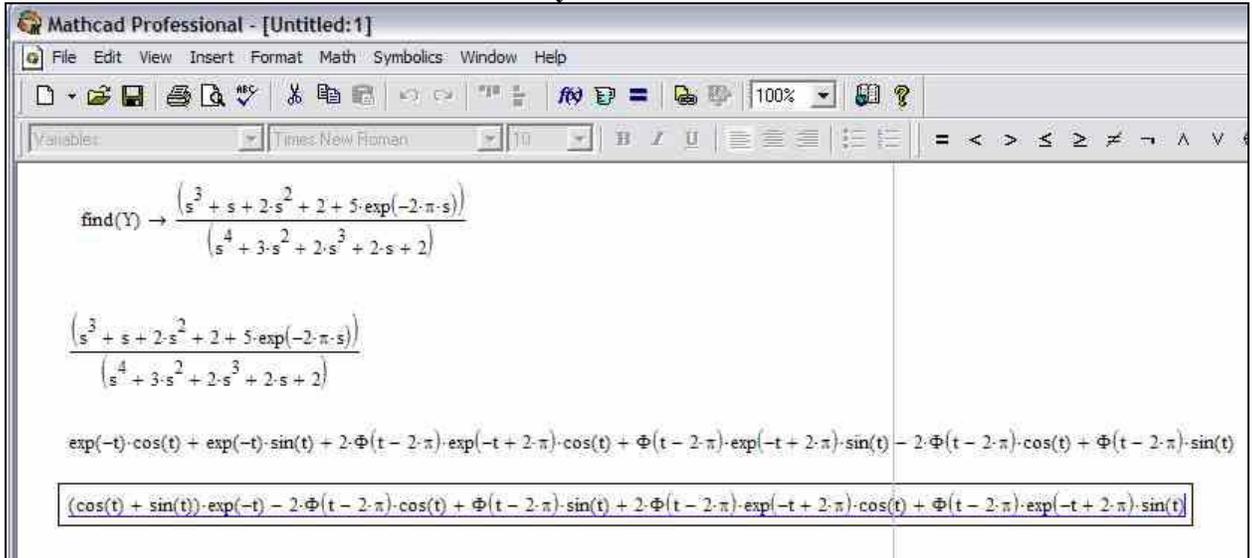
**Rearrange the Answer**

11. You will now rearrange the answer into a more standard form by collecting terms. Highlight the “exp(-t)” term come so that it is shown in reverse video as shown in figure 11. Then choose "Symbolics/Collect" from the menu. Your worksheet should resemble figure 11 a. After you click the menu, Mathcad will work a few seconds and collect all the terms containing the term you have highlighted. After collecting terms, the result will be displayed under the original function as shown in figure 11b.

**Figure 11a**  
**Choose "Symbolics/Collect" to Simplify the Function**

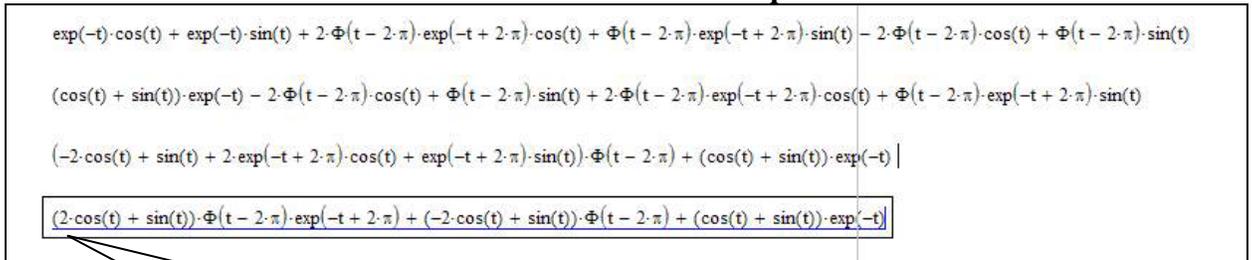
The screenshot shows the Mathcad Professional interface. The menu bar includes File, Edit, View, Insert, Format, Math, Symbolics, Window, and Help. The 'Symbolics' menu is open, with 'Collect' highlighted. The main workspace contains the same mathematical expression as in Figure 10, with the exponential term  $\exp(-t)$  highlighted in reverse video. The bottom of the window shows the result of the 'Collect' operation, which is the same as the result in Figure 10.

**Figure 11b**  
**Results of Mathcad's "Symbolics/Collect" Function**



12. Complete the simplification by collecting the  $\Phi(t - 2\pi)$  terms using the "Symbolics/Collect" function. Finally collect the  $\exp(-2+2\pi)$  terms the same way you have collected the other terms. After these two operations your worksheet should resemble figure 12, which is a simplified solution to the differential equation.

**Figure 12**  
**Final Solution of the Differential Equation**



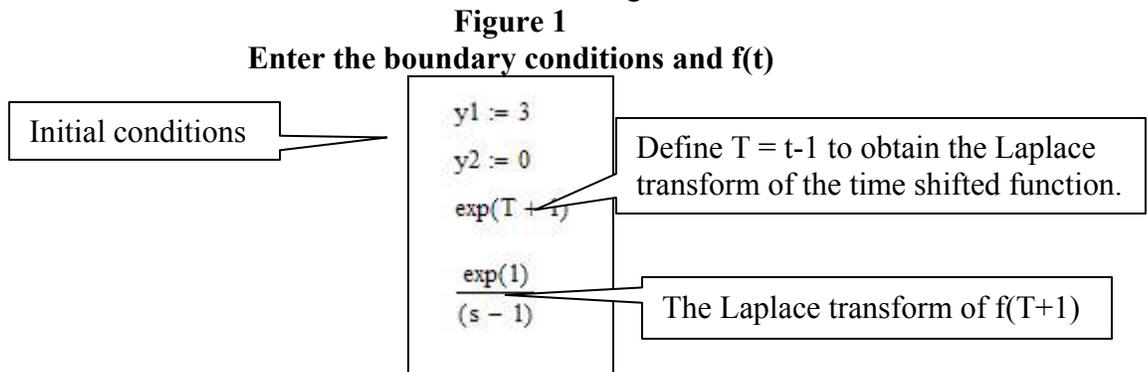
Final result after collecting all terms.

## II. Solving a System of Ordinary Differential Equations

1. You are now ready to solve the following linear system of ordinary differential equations using Laplace transforms.

$$\begin{aligned} y_1' &= 2y_1 - 4y_2 + u(t-1)e^t \\ y_2' &= y_1 - 3y_2 + u(t-1)e^t \end{aligned}$$

with the boundary conditions  $y_1(0) = 3, y_2(0) = 0$  (Advanced Engineering Mathematics eight edition, Erwin Kreyszig, problem set 5.6, no. 13, page 295). Begin by entering the boundary conditions. Time shift the exponential function by defining  $T = t - 1$  so  $t = T+1$ . Therefore, the function you are going to convert to Laplace transform is  $\exp(T+1)$ . Enter this function below the boundary conditions, highlight the “T” and choose “Symbolics/Transform/Laplace” from the menu. Your worksheet should look like Figure 1.



2. Now, set up a solve block containing the Laplace transforms of the differential equation shot of the top of page. Using the variables already defined for the boundary conditions, these equations become:

$$\begin{aligned} sY_1 - y_1 &= 2 * Y_1 - 4 * Y_2 + F(s) \\ sY_2 - y_2 &= Y_1 - 3 * Y_2 + F(s) \end{aligned}$$

Where  $F(s) =$  Laplace transform of  $u(t-a) e^t$  which is  $e^{-s}$  time the Laplace

transform of  $e^{T+1}$ , which you already found to be  $\frac{\exp(1)}{s - 1}$  using Mathcad's

Laplace transform function.

Type the word “given”. Underneath this word and are the equations representing old Laplace transforms of the system of differential equations. Remember to use the “=” symbol in the Boolean toolbar. Also, remember to replace  $F(s)$  with

$\exp(-s) \cdot \frac{\exp(1)}{(s - 1)}$ . You may find it advantageous to copy and paste the Laplace

transform. When you have entered these items your worksheet should look like figure 2.

**Figure 2**  
**Enter the Laplace Transforms for the System of Differential Equations**

$$\begin{aligned}
 &y_1 := 3 \\
 &y_2 := 0 \\
 &\exp(T + 1) \\
 &\frac{\exp(1)}{(s - 1)} \\
 &\text{Given} \\
 &s \cdot Y_1 - y_1 = 2 \cdot Y_1 - 4 \cdot Y_2 + \exp(-s) \cdot \frac{\exp(1)}{(s - 1)} \\
 &\boxed{s \cdot Y_2 - y_2 = Y_1 - 3 \cdot Y_2 + \exp(-s) \cdot \frac{\exp(1)}{(s - 1)}}
 \end{aligned}$$

3. Now, have Mathcad solve the system of equations algebraically for Y1 and Y2. Under the last equation type “find(Y1, Y2)” and use the arrow button in the evaluation toolbar. The Laplace domain solution to the differential equations should appear to the right of the arrow operator. As before, Mathcad automatically substituted in the numerical values for the boundary conditions. Your worksheet should resemble figure 3.

**Figure 3**  
**Mathcad Solution to the Laplace Transforms**

Given

$$s \cdot Y_1 - y_1 = 2 \cdot Y_1 - 4 \cdot Y_2 + \exp(-s) \cdot \frac{\exp(1)}{(s-1)}$$

$$s \cdot Y_2 - y_2 = Y_1 - 3 \cdot Y_2 + \exp(-s) \cdot \frac{\exp(1)}{(s-1)}$$

Find(Y1, Y2) →

$$\begin{bmatrix} \frac{(3 \cdot s \cdot \exp(s) + 9 \cdot \exp(s) + \exp(1))}{\exp(s) \cdot (s-1) \cdot (s+2)} \\ \frac{(3 \cdot \exp(s) + \exp(1))}{\exp(s) \cdot (s^2 + s - 2)} \end{bmatrix}$$

4. The solutions to the system of equations are given in the order of the variables in the “find” command. Therefore, the function for Y1 is the first, or top, function. You can return this function to the time domain as you did in the previous example: copy the function below the find block, highlight any letter “s” and then choose “Symbolics/Transform/Inverse Laplace” from the menu. Follow the steps in "Converting Back to the Time Domain" in the previous section. The resulting transform it is shown in figure 4

**Figure 4**  
**Copy the Laplace Transform of Y1(S) and Obtain y1(T)**

Find(Y1, Y2) →

$$\begin{bmatrix} \frac{(3 \cdot s \cdot \exp(s) + 9 \cdot \exp(s) + \exp(1))}{\exp(s) \cdot (s-1) \cdot (s+2)} \\ \frac{(3 \cdot \exp(s) + \exp(1))}{\exp(s) \cdot (s^2 + s - 2)} \end{bmatrix}$$

$$\frac{(3 \cdot s \cdot \exp(s) + 9 \cdot \exp(s) + \exp(1))}{\exp(s) \cdot (s-1) \cdot (s+2)}$$

$$4 \cdot \exp(t) - \exp(-2 \cdot t) + \frac{1}{3} \cdot \Phi(t-1) \cdot \exp(t) - \frac{1}{3} \cdot \Phi(t-1) \cdot \exp(3-2 \cdot t)$$

5. The function for  $y_2(t)$  is obtained in a similar manner. Again, refer back to "Converting Back to the Time Domain" in the previous section to get the details. The resulting transformation is shown in figure 5.

**Figure 5**  
**Transforming  $Y_2(s)$  to  $y_2(t)$**

$$\frac{(3 \cdot \exp(s) + \exp(1))}{\exp(s) \cdot (s^2 + s - 2)}$$
$$\exp(t) - \exp(-2 \cdot t) + \frac{1}{3} \cdot \Phi(t - 1) \cdot \exp(t) - \frac{1}{3} \cdot \Phi(t - 1) \cdot \exp(3 - 2 \cdot t)$$