Numerical Solutions of Ordinary Differential Equations
Charles Nippert

This set of notes will describe one of several methods that can be used to solve ordinary
differential equations. As an example you will solve the second order differential equation
\[ \frac{d^2y}{dt^2} + y = 2x \]
with the boundary conditions
\[ y(0) = -1 \]
\[ y'(0) = 8 \]
The result of this numerical method will be a Mathcad function that you will use to draw a graph
and display a table of results.

Open Mathcad as you do normally. Mathcad has several convenient methods of solving
numerically ordinary differential equations. The syntax of these methods varies as does the
computational procedure "under the hood". The simplest and most general method involves the
use of the Mathcad function “odesolve”, which you will use in this example.
Enter the differential equation

1. The ordinary differential equation will be contained in a "block" that is separated from the rest of the worksheet at both the top and bottom so that Mathcad knows where the differential equation and boundary conditions are. The top of the block is marked by the word "given". This word is typed directly on the worksheet and is not entered as a "comment" or text. Press "Enter" after you have typed “given” and use the arrow keys to move the cursor a little lower and slightly to the right. This last movement of the cursor is merely to improve the appearance of the finished worksheet. Open the calculus toolbar by choosing "View/Toolbars/Calculus" from the menu. Enter the left-hand side of the differential equation. Begin by pressing the button \( \frac{d}{dx} \) on the Evaluation toolbar to create a higher order derivative. Mathcad requires that the variable being differentiated be expressed as a function. Type "y(x)" for the numerator. Use the arrow keys to move the cursor to one of the placeholders for the order. Type “2” for the order of the derivative and, "x" for the denominator of the derivative. When you finish entering the derivative, use the arrow keys to move the blue cursor to the right. The cursor should enclose the entire derivative as shown in figure. If the blue cursor disappears and the red cross reappears, you have moved the cursor into an empty portion of the worksheet. Use the arrow keys to get "back into" the function you are typing. Your screen should now look like figure 1.

Figure 1
After Step 1
2. Enter the rest of the left-hand side of the differential equation by typing "+y(t)". If the "Boolean" dialog box is not open, open it now by choosing "View/Toolbars/Boolean" from the menu. Press the boldface "=" button on the Boolean dialog box. A "=" will appear. Note that this is a different symbol than is used for the arithmetic assignment statement used in earlier notes. This symbol is used to identify equations in the “solve blocks” which Mathcad uses for solving differential equations, simultaneous equations, linear programming problems and, other mathematical operations. Type "2*x" to finish this equation. Your sheet should now look like figure 2.

Figure 2
After Space Step 2
3. Now, you will enter the boundary conditions. Press "Enter". To enter the value of y at t = 0 you will type "y(0)"; press the "=" button on the Boolean dialog box and, type "-1". Note that the variable y is represented as a function and the value of the parameter x is provided in the function. Your screen will now look like figure 3.

![Figure 3](image)

After Step 3

4. Mathcad requires that the boundary conditions such as \( \frac{dy(x)}{dx} \bigg|_{x=0} = 8 \) use the apostrophe notation (that is, y’ to represent dy/dx) for derivatives. This difference in notation provides a distinction between statements that describe boundary conditions and statements that describe differential equations. Press "Enter" to begin entering this boundary condition just under the last one. Type the character “y'”. Next press "Ctrl F7". An ’ will appear just after the “y”. The number of apostrophes it is the order of the boundary condition. Finish entering the boundary condition by typing “(0)”, pressing the boldface "=" button and typing "8". Your worksheet should now look like figure 4.
5. You will now finish the creation of the program to find the numerical solution of this differential equation by providing Mathcad with the name of a function that you can use in later portions of the sheet to generate numerical answers within the integration interval you will provide. The name you will give this function is "ya". The "a" in this variable name is a little lower than the rest of the letters in the name. However this "a" is not the index of an array. Mathcad provides you with a method of showing subscripts that are not parts of arrays. These are called "decorative" subscripts because their use is merely to enhance the appearance of the worksheet. Press "Enter" to leave the current statement and type "y". The portion of your screen where you are entering the equation should look like figure 5a on the left. Now press the "." key. The cursor should change to figure 5b. The inverted blue "L" becomes a little larger and moves down and to the right. Compare this figure to figure 5c which shows the screen when entering the subscript of an array. Press the "a:" keys. The decorative subscript mode will continue until an operator key, such as "+", "-", "*", "/" or ":" (for the = ) is pressed. Your screen should now look like figure 5d.

<table>
<thead>
<tr>
<th>Figure 5a</th>
<th>Figure 5b</th>
<th>Figure 5c</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cursor Position for an Ordinary Variable Name</td>
<td>Cursor when typing a &quot;Decorative&quot; subscript</td>
<td>Cursor and black rectangle when entering an index of an array</td>
</tr>
<tr>
<td>y</td>
<td>y</td>
<td>y</td>
</tr>
</tbody>
</table>
6. The last step involves specifying what function Mathcad is to use when processing the solve block you have just entered. The function you will use for this problem is named "Odesolve". This is a general-purpose solver for both initial value and boundary value problems. "Odesolve" takes two arguments; the first is the name of the integral variable, “x” in this case and, the second argument is the upper bond of the integration. This second value must be greater than the value of the parameter in any of the boundary conditions. Enter "Odesolve(x, 10)". This statement will create a function named “ya” that will return the value of the solution of this differential equation for any x between zero and 10.
7. You will now plot the function $y_a$ versus $x$. Press "Enter" to leave the state menu at just entered, then choose "Insert/Graph/X-Y Plot". After a few seconds, the shape of the next X-Y plot will appear as two rectangles, one inside the other. A portion of the screen is shown below in figure 7. The smaller solid black rectangles attached to the outer rectangle are "handles" that you can use to move or stretch the graph. The two solid black rectangles just inside the outer rectangle are areas where you will enter the names of the axes.

To illustrate how the handles work to illustrate how the handles work, move the mouse cursor over one of the small black rectangles. Notice that when the cursor is over the rectangle, the cursor first turns to the shape of a hand then to the shape of a double headed arrow. When the cursor is in the shape of a hand, moving the cursor while pressing and holding the left mouse button will drag the graph to a new location. When the cursor is in the shape of a double and headed arrow, moving the cursor while pressing and holding the left mouse button will change the shape of the graph.

Adding Variables to the Plot

8. Note that the blue inverted “L” cursor is on the solid black rectangle indicating the name of the variable used in the horizontal axis. Press “x”. The letter “x” should replace the black rectangle on the horizontal axis. Two solid black rectangles should appear at either end of the smaller rectangle representing the plot. These two black rectangles can be used and are the upper and lower values of the horizontal axis. For now you will use Mathcad auto scaling function to determine these values. Your graph should now look like figure 8.
There are several ways to move the cursor to the small black rectangle, or placeholder, where you'll enter the variable to be plotted on the vertical axis. Use either the arrow keys on the keyboard or move the mouse cursor over that rectangle and left click. Either method will work. Use the method that you find more convenient. Type the characters “\( y_a(x) \)”. Note that you are using the variable name “\( y_a \)” as a function of “\( x \)”. Press "Enter" to draw the graph. Your screen should not look like figure 9. The horizontal axis will range from the initial conditions to the variable specified in the function definition. The vertical axis will be scaled automatically based on the values returned for \( y_a \).
Formatting Your Graph (optional)

10. You can change the appearance of the graph to improve your presentation. Move the mouse cursor over the graph and click the left mouse button once. The outer boundary rectangle for the graph will appear. Next move the cursor over the small solid black rectangle in the lower right hand corner. The cursor will turn into the shape of a double headed arrow. Press and hold the left mouse button while moving the cursor to change the size and shape of your graph. Drag the cursor down into the right to enlarge and the graph. Next, double-click on the graph to call up the plot dialog box. Your worksheet should now look something like figure 10.

Figure 10
After Step 10

11. Check the boxes marked "Grid Lines" for the x-axis and y-axis. Then choose the tab labeled "Traces". The traces dialogue allows you to specify the way the line is drawn. Click on "Trace 1" and change the color to blue and the line width to 2. The dialog box should look like figure 11. The graph will not change until you are finished and close the dialog box by pressing the OK button after you have added labels.
12. Click on the labels tab. You can now enter labels for the axes and a title for the graph. Enter the title for the graph and the axes in their respective fields. Make sure to check the "Show Title" box underneath the title of the graph. The dialog box should look like figure 12.

Press “OK” when finished.
13. You can close the dialog box by pressing "OK". When the dialog box is closed to change is in the formatting will appear on the graph. Your graph should now look like figure 13.

Figure 13
After Formatting Your Graph